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The application of the $\overline{\partial}$-dressing method to some integrable $(2+1)$-dimensional nonlinear equations

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Abstract. Some integrable (2 + 1)-dimensional nonlinear equations which are the generalizations of dispersive long wave, nonlinear Schrödinger, sinh–Gordon and heat equations are studied by the inverse spectral transform method. The solutions with functional parameters, line solitons and rational solutions of these equations are constructed via $\bar{\partial}$-dressing method.

1. Introduction

It is well known now that the $\bar{\partial}$-dressing method is very powerful, fundamental and, at the same time, simple method for the construction of exact solutions of $(1+1)$-dimensional as well as $(2+1)$-dimensional integrable nonlinear equations. The essentials of this method have been developed in the papers of Zakharov and Manakov [1, 2], see also the papers [3–5] and books [6, 7].

In the present paper we consider some integrable $(2+1)$-dimensional generalizations of dispersive long wave, nonlinear Schrödinger, sinh–Gordon and heat equations. All these equations are well known, and several classes of their exact solutions have been constructed by different means. Our goal is to show how $\bar{\partial}$-dressing method can be applied systematically for the construction of broad classes of exact solutions of the above mentioned equations. The results that we have obtained in this way are not completely new and certainly overlap with (or reproduce) the results of other investigations in the framework of other approaches, but we believe that the application of the $\bar{\partial}$-dressing method even for known cases of integrable equations may be instructive and useful.

Let us start from the following two linear auxiliary problems [8]:

$$
\begin{aligned}
&L_1\Psi = \Psi_{\xi\eta} + V\Psi_\eta + U\Psi = 0\\
&L_2\Psi = \Psi_t + \alpha\Psi_{\xi\xi} + \beta\Psi_{\eta\eta} + W_1\Psi_\eta + W_2\Psi = 0
\end{aligned}
\tag{1}
$$

where α, β are constants; $\xi := x - \sigma y$, $\eta := x + \sigma y$, $\sigma^2 = 1$. The compatibility condition for system (1) is the triad operator representation of the form [8]:

$$
[L_1, L_2] = (W_{1\eta} - 2\alpha V_\xi)L_1. \tag{2}
$$

Here $W_1(\xi,\eta,t) = 2\beta\partial_\xi^{-1}V_\eta$, $W_2(\xi,\eta,t) = 2\alpha\partial_\eta^{-1}U_\xi$ and the system of nonlinear equations which is integrable by (1) is the following:

$$
\begin{aligned}
&U_t - \alpha U_{\xi\xi} + \beta U_{\eta\eta} - 2\alpha(UV)_\xi + 2\beta(U\partial_\xi^{-1}V_\eta)_\eta = 0\\
&V_t + \alpha V_{\xi\xi} - \beta V_{\eta\eta} + 2\beta U_\eta - \alpha(V^2)_\xi - 2\alpha\partial_\eta^{-1}U_{\xi\xi} + 2\beta V_\eta\partial_\xi^{-1}V_\eta = 0.
\end{aligned}
\tag{3}
$$

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For $\alpha = 1$, $\beta = 0$ system (3) has the form

$$U_t - U_{\xi\xi} - 2(UV)_\xi = 0$$
$$V_{t\eta} + V_{\xi\xi\eta} - 2U_{\xi\xi} - (V^2)_{\xi\eta} = 0 \tag{4}$$

which is known as an integrable $(2+1)$-dimensional generalization of dispersive long-wave system [9]. On the introduction of the new dependent variable $\varphi = \ln 4U$ and by appropriate elimination of another variable V by the formula $V = (1/2)(e^{-\varphi}\partial_\xi^{-1}(e^\varphi \varphi_t) - \varphi_\xi)$ system (4) is reduced to a single equation for the φ-2D sinh–Gordon equation [9]:

$$(e^{-\varphi}[e^\varphi(\varphi_{\xi\eta} + \sinh\varphi)]_\xi)_\xi - (e^{-\varphi}\partial_\xi^{-1}(e^\varphi)_t)_{t\eta} + \tfrac{1}{2}[(e^{-\varphi}\partial_\xi^{-1}(e^\varphi)_t)^2]_{\xi\eta} = 0 \tag{5}$$

some $(2+1)$-dimensional generalization (non-symmetrical in ξ and η) of sinh–Gordon equation $\varphi_{\xi\eta} + \sinh\varphi = 0$. Let us note that equation (5) coincides with the corresponding 2D sinh–Gordon equation of the paper of Boiti *et al* [9] under the following identification of independent variables: ξ with x, η with t and t with y.

System (3) also admits the reductions $U = 0$, $V = Q_\xi$ and $U = Q_{\xi\eta}$, $V = Q_\xi$ to the equations

$$Q_t \pm (\alpha Q_{\xi\xi} - \beta Q_{\eta\eta}) - \alpha Q_\xi^2 + \beta Q_\eta^2 = 0 \tag{6}$$

with the upper sign for the first and the lower sign for the second reduction. Equations (6) are the $(2+1)$-dimensional generalizations of Burgers equation and under the transformations $Q = \mp \ln q$ reduce to linear equations $q_t \pm (\alpha q_{\xi\xi} - \beta q_{\eta\eta}) = 0$.

Under the change of dependent variables

$$V = -q_\xi/q \qquad U = -pq \tag{7}$$

system (3) reduces to the Davey–Stewartson (DS) system of equations:

$$q_t + \alpha q_{\xi\xi} - \beta q_{\eta\eta} - 2\alpha q \partial_\eta^{-1}(pq)_\xi + 2\beta q \partial_\xi^{-1}(pq)_\eta = 0$$
$$p_t - \alpha p_{\xi\xi} + \beta p_{\eta\eta} + 2\alpha p \partial_\eta^{-1}(pq)_\xi - 2\beta p \partial_\xi^{-1}(pq)_\eta = 0. \tag{8}$$

Different choices for α and β correspond to DS-1, DS-2, ...systems of equations. When α and β are pure imaginary constants system (8) admits the reduction $p = \kappa\bar{q}$ to the single DS equation:

$$q_t + \alpha q_{\xi\xi} - \beta q_{\eta\eta} - 2\alpha\kappa q \partial_\eta^{-1}(|q|^2)_\xi + 2\beta\kappa q \partial_\xi^{-1}(|q|^2)_\eta = 0. \tag{9}$$

The particular case $\beta = 0$ of system (8) may also be interesting:

$$q_t + \alpha q_{\xi\xi} - 2\alpha q \partial_\eta^{-1}(pq)_\xi = 0$$
$$p_t - \alpha p_{\xi\xi} + 2\alpha p \partial_\eta^{-1}(pq)_\xi = 0 \tag{10}$$

with the corresponding reduction $p = \kappa\bar{q}$ in the case $\bar{\alpha} = -\alpha$ [10].

In the following sections we apply the $\bar\partial$-dressing method to the construction of broad classes of exact solutions of all the above mentioned equations (3)–(5) and (8)–(10) under the assumption that U in (1) has a constant asymptotic value at infinity. The paper is organized as follows. In section 2 the basic ingredients of $\bar\partial$-dressing method are considered. The classes of exact solutions with functional parameters, line solitons and rational solutions are presented in sections 3, 4 and 5, respectively.

2. Basic ingredients of the $\bar\partial$-dressing method

Let us apply the $\bar\partial$-dressing method [1–5] for system (1) in the case when $U(\xi,\eta,t)$ has a generically non-zero asymptotic value $U_\infty = -\epsilon$ at infinity:

$$U(\xi,\eta,t) = \tilde U(\xi,\eta,t) + U_\infty = \tilde U(\xi,\eta,t) - \epsilon \tag{11}$$

where $\tilde U(\xi,\eta,t) \to 0$ as $\xi^2+\eta^2\to\infty$. At first one postulates non-local $\bar\partial$-problem:

$$\frac{\partial\chi(\lambda,\bar\lambda)}{\partial\bar\lambda} = (\chi * R)(\lambda,\bar\lambda) = \iint_C \frac{\mathrm{d}\lambda'\wedge \mathrm{d}\overline{\lambda'}}{2\pi\mathrm{i}}\chi(\lambda',\overline{\lambda'})R(\lambda',\overline{\lambda'};\lambda,\bar\lambda). \tag{12}$$

The functions χ and R in our case are the scalar complex-valued functions. For function χ we choose the canonical normalization ($\chi\to 1$, as $\lambda\to\infty$). We assume also that problem (12) is uniquely solvable.

Then one introduces the dependence of kernel R on space and time variables ξ, η, t:

$$\begin{aligned}
\frac{\partial R}{\partial\xi} &= \mathrm{i}\lambda' R(\lambda',\overline{\lambda'};\lambda,\bar\lambda;\xi,\eta,t) - R(\lambda',\overline{\lambda'};\lambda,\bar\lambda;\xi,\eta,t)\mathrm{i}\lambda\\
\frac{\partial R}{\partial\eta} &= -\frac{\mathrm{i}\epsilon}{\lambda'}R(\lambda',\overline{\lambda'};\lambda,\bar\lambda;\xi,\eta,t) + R(\lambda',\overline{\lambda'};\lambda,\bar\lambda;\xi,\eta,t)\frac{\mathrm{i}\epsilon}{\lambda}\\
\frac{\partial R}{\partial t} &= \left(\alpha\lambda'^2+\frac{\beta\epsilon^2}{\lambda'^2}\right)R(\lambda',\overline{\lambda'};\lambda,\bar\lambda;\xi,\eta,t) - R(\lambda',\overline{\lambda'};\lambda,\bar\lambda;\xi,\eta,t)\left(\alpha\lambda^2+\frac{\beta\epsilon^2}{\lambda^2}\right)
\end{aligned} \tag{13}$$

i.e.

$$R(\lambda',\overline{\lambda'};\lambda,\bar\lambda;\xi,\eta,t) = R_0(\lambda',\overline{\lambda'};\lambda,\bar\lambda)\exp(F(\lambda')-F(\lambda)) \tag{14}$$

where

$$F(\lambda) := \mathrm{i}\left(\lambda\xi - \frac{\epsilon}{\lambda}\eta\right) + \left(\alpha\lambda^2+\frac{\beta\epsilon^2}{\lambda^2}\right)t. \tag{15}$$

With the use of 'long' derivatives

$$D_\xi = \partial_\xi + i\lambda \qquad D_\eta = \partial_\eta - \frac{\mathrm{i}\epsilon}{\lambda} \qquad D_t = \partial_t + \alpha\lambda^2 + \frac{\beta\epsilon^2}{\lambda^2} \tag{16}$$

the dependence on ξ, η, t can be expressed in the form

$$[D_\xi, R] = 0 \qquad [D_\eta, R] = 0 \qquad [D_t, R] = 0. \tag{17}$$

By the use of derivatives (16) one then constructs linear operators

$$L = \sum u_{lmn}(\xi,\eta,t) D_\xi^l D_\eta^m D_t^n \tag{18}$$

which satisfy the condition

$$\left[\frac{\partial}{\partial\bar\lambda}, L\right] = 0 \tag{19}$$

in the absence of singularities on λ. For such operators L the function $L\chi$ obeys the same $\bar\partial$-equation as the function χ. If there are several operators L_i of this type then by virtue of the unique solvability of (12) one has $L_i\chi = 0$. In our case one can construct two such operators:

$$\begin{aligned}
L_1\chi &= (D_\xi D_\eta + \tilde V D_\xi + V D_\eta + U)\chi = 0\\
L_2\chi &= (D_t + \alpha D_\xi^2 + \beta D_\eta^2 + \tilde W_1 D_\xi + W_1 D_\eta + W_2)\chi = 0.
\end{aligned} \tag{20}$$

Indeed let us consider (20) for the series expansion of χ near points $\lambda = 0$ and $\lambda = \infty$: $\chi = \tilde{\chi}_0 + \lambda\chi_1 + \lambda^2\chi_2 + \cdots$, $\chi = \chi_0 + \chi_{-1}/\lambda + \chi_{-2}/(\lambda^2) + \cdots$. In the neighbourhood of $\lambda = \infty$, equating to zero the coefficients for degrees of λ, we obtain from $L_1\chi = 0$:

$$\begin{aligned} &\lambda:\ \mathrm{i}\chi_{0\eta} + \mathrm{i}\tilde{V}\chi_0 = 0 \\ &\lambda^0:\ \chi_{0\xi\eta} + \tilde{V}\chi_{0\xi} + V\chi_{0\eta} + \mathrm{i}\chi_{-1\eta} + \mathrm{i}\tilde{V}\chi_{-1} + (U+\epsilon)\chi_0 = 0 \end{aligned} \tag{21}$$

and from $L_2\chi = 0$:

$$\begin{aligned} &\lambda:\ 2\mathrm{i}\alpha\chi_{0\xi} + \mathrm{i}\tilde{W}_1\chi_0 = 0 \\ &\lambda^0:\ 2\mathrm{i}\alpha\chi_{-1\xi} + \tilde{W}_1\chi_{0\xi} + W_1\chi_{0\eta} + W_2\chi_0 = 0. \end{aligned} \tag{22}$$

Analogously, in the neighbourhood of $\lambda = 0$, from $L_1\chi = 0$:

$$\lambda^{-1}:\ -\mathrm{i}\tilde{\chi}_{0\xi} - \mathrm{i}V\tilde{\chi}_0 = 0 \tag{23}$$

and from $L_2\chi = 0$:

$$\lambda^{-1}:\ -2\mathrm{i}\beta\tilde{\chi}_{0\eta} - \mathrm{i}W_1\tilde{\chi}_0 = 0. \tag{24}$$

Due to canonical normalization $\chi_0 = 1$ and from (21), (22) it follows for $\tilde{V}$, and $\tilde{W}_1$: $\tilde{V} = 0$, $\tilde{W}_1 = 0$. Then from (21)–(24) we obtain for V, U, W_1 and W_2 the following reconstruction formulae:

$$\begin{aligned} &V = -\tilde{\chi}_{0\xi}/\tilde{\chi}_0 \qquad U = -\epsilon - \mathrm{i}\chi_{-1\eta} \\ &W_1 = -2\beta\tilde{\chi}_{0\eta}/\tilde{\chi}_0 \qquad W_2 = -2\mathrm{i}\alpha\chi_{-1\xi}. \end{aligned} \tag{25}$$

In terms of wavefunction

$$\psi := \chi \exp\left[\mathrm{i}\left(\lambda\xi - \frac{\beta\eta}{\lambda}\right) + \left(\alpha\lambda^2 + \frac{\beta\epsilon^2}{\lambda^2}\right)t\right]$$

one obtains from (20) our auxiliary problem (1).

The solution of $\bar{\partial}$-problem (12) with the canonical normalization $\chi_0 = 1$ is equivalent to the solution of the following singular integral equation:

$$\chi(\lambda) = 1 + \iint_C \frac{\mathrm{d}\lambda' \wedge \mathrm{d}\overline{\lambda'}}{2\pi\mathrm{i}(\lambda' - \lambda)} \iint_C \frac{\mathrm{d}\mu \wedge \mathrm{d}\overline{\mu}}{2\pi\mathrm{i}} \chi(\mu, \overline{\mu}) R_0(\mu, \overline{\mu}; \lambda', \overline{\lambda'}) \mathrm{e}^{(F(\mu) - F(\lambda'))}. \tag{26}$$

From (26) one obtains for the coefficients $\tilde{\chi}_0$ and χ_{-1} of the series expansion of χ:

$$\begin{aligned} &\tilde{\chi}_0 = 1 + \iint_C \frac{\mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda}}{2\pi\mathrm{i}\lambda} \iint_C \frac{\mathrm{d}\mu \wedge \mathrm{d}\overline{\mu}}{2\pi\mathrm{i}} \chi(\mu, \overline{\mu}) R_0(\mu, \overline{\mu}; \lambda, \bar{\lambda}) \mathrm{e}^{(F(\mu) - F(\lambda))} \\ &\chi_{-1} = -\iint_C \frac{\mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda}}{2\pi\mathrm{i}} \iint_C \frac{\mathrm{d}\mu \wedge \mathrm{d}\overline{\mu}}{2\pi\mathrm{i}} \chi(\mu, \overline{\mu}) R_0(\mu, \overline{\mu}; \lambda, \bar{\lambda}) \mathrm{e}^{(F(\mu) - F(\lambda))} \end{aligned} \tag{27}$$

where $F(\lambda)$ is given by the formula (15).

In conclusion of this section let us consider the conditions of the reality of U and V. One must distinguish two different cases. For real values of α and β the condition of reality of U, V leads from (25) and (27) in the limit of weak fields to the following restriction on the kernel R of $\bar{\partial}$-problem:

$$\overline{R_0(\mu, \overline{\mu}; \lambda, \bar{\lambda})} = -R_0(-\overline{\mu}, -\mu; -\bar{\lambda}, -\lambda). \tag{28}$$

In the case of pure imaginary values of α and β the condition of the reality of U leads from (25), (27) in the limit of weak fields to another restriction on the kernel R of $\bar{\partial}$-problem:

$$\overline{R_0(\mu, \overline{\mu}; \lambda, \bar{\lambda})} = -R_0(\bar{\lambda}, \lambda; \overline{\mu}, \mu). \tag{29}$$

In terms of variables p and q (7) the condition of reality of $U = -pq$ means that $p = \kappa\bar{q}$. Different choices for the kernel R of $\bar{\partial}$-problem (12) satisfying to restrictions (28), (29) lead to different classes of exact solutions of integrable nonlinear equations (3)–(5) and (8)–(10).

3. The solutions with functional parameters

At first let us consider general class of exact solutions of equations (3)–(5) and (8)–(10) which corresponds to the following degenerate kernel R_0 of $\bar{\partial}$-problem:

$$R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \mathrm{i} \sum_{k=1}^{N} f_k(\mu, \bar{\mu}) g_k(\lambda, \bar{\lambda}). \tag{30}$$

For the kernel of this type equation (26) gives

$$\chi(\lambda, \bar{\lambda}) = 1 + \mathrm{i} \sum_{k=1}^{N} h_k(\xi, \eta, t) \iint_C \frac{\mathrm{d}\lambda' \wedge \mathrm{d}\bar{\lambda'}}{2\pi \mathrm{i}(\lambda' - \lambda)} g_k(\lambda', \bar{\lambda'}) \mathrm{e}^{-F(\lambda')} \tag{31}$$

where

$$h_k(\xi, \eta, t) := \iint_C \frac{\mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda}}{2\pi \mathrm{i}} f_k(\lambda, \bar{\lambda}) \chi(\lambda, \bar{\lambda}) \mathrm{e}^{F(\lambda)} \tag{32}$$

and $F(\lambda)$ is defined in (15). The quantities h_k can be calculated from the algebraic system:

$$\sum_{l=1}^{N} A_{kl} h_l = \xi_k \qquad (k = 1, \ldots, N) \tag{33}$$

where

$$\xi_k(\xi, \eta, t) := \iint_C \frac{\mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda}}{2\pi \mathrm{i}} f_k(\lambda, \bar{\lambda}) \mathrm{e}^{F(\lambda)} \tag{34}$$

and

$$A_{kl} := \delta_{kl} - \mathrm{i} \iint_C \frac{\mathrm{d}\lambda \wedge \bar{\lambda}}{2\pi \mathrm{i}} \iint_C \frac{\mathrm{d}\lambda' \wedge \mathrm{d}\bar{\lambda'}}{2\pi \mathrm{i}(\lambda' - \lambda)} \mathrm{e}^{F(\lambda) - F(\lambda')} f_k(\lambda, \bar{\lambda}) g_l(\lambda', \bar{\lambda'}). \tag{35}$$

System (33) arises from (31) after multiplication by $\mathrm{e}^{F(\lambda)} f(\lambda, \bar{\lambda})$ and integration over λ.

Solving system (33) for arbitrary given functions f_l and g_l, one finds for $\tilde{\chi}_0$ and χ_{-1} given by (27) the following expressions:

$$\begin{aligned} \tilde{\chi}_0 &= 1 + \sum_{k=1}^{N} h_k \eta_k = 1 + \sum_{k=1, l=1}^{N} \eta_k A_{kl}^{-1} \xi_l \\ \chi_{-1} &= -\mathrm{i} \sum_{k=1}^{N} h_k \eta_{k\xi} = -\mathrm{i} \sum_{k=1, l=1}^{N} \eta_{k\xi} A_{kl}^{-1} \xi_l \end{aligned} \tag{36}$$

where

$$\eta_k(\xi, \eta, t) := \iint_C \frac{\mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda}}{2\pi \mathrm{i}} g_k(\lambda, \bar{\lambda}) \mathrm{e}^{-F(\lambda)}. \tag{37}$$

Using (35) and the definitions (34), (37) for ξ_k and η_k one can show that

$$A_{kl} = \delta_{kl} - \partial_\eta^{-1}(\xi_{k\eta} \eta_l). \tag{38}$$

Then by using the formulae of reconstruction (25) and (36) one obtains for the exact solutions V, U of systems (3), (4) and φ of 2D sinh–Gordon equation (5) the following expressions:

$$\begin{aligned} V &= -\frac{\partial}{\partial \xi} \ln \det(1 + M) \\ U &= -\epsilon \det[(1 + M)(1 + \tilde{M})] \qquad \varphi = \ln 4U|_{\alpha=1, \beta=0} \end{aligned} \tag{39}$$

where

$$M_{kn} := \sum_{l=1}^{N} \eta_k A_{nl}^{-1} \xi_l \qquad \tilde{M}_{kn} := \frac{1}{\epsilon} \sum_{l=1}^{N} \eta_{k\xi} A_{nl}^{-1} \xi_{l\eta}. \tag{40}$$

The simplest example of such solutions with functional parameters of equations (4), (5) was constructed for the first time by another method in the paper by Boiti *et al* [9].

The solutions q, p of systems (8), (10) according (7), (39) have the form:

$$q = \det(1 + M) \qquad p = \epsilon \det(1 + \tilde{M}). \tag{41}$$

The reality conditions (28) and (29) imply certain constraints on the functions f_k and g_k $(k = 1, \ldots, N)$. They are satisfied, in particular, if

$$\overline{f_k(\mu, \overline{\mu})} = f_k(-\overline{\mu}, -\mu), \qquad \overline{g_k(\mu, \overline{\mu})} = g_k(-\overline{\mu}, -\mu) \tag{42}$$

for the case of real values of α, β, and

$$\overline{f_k(\mu, \overline{\mu})} = R_k g_k(\overline{\mu}, \mu) \qquad (R_k = \overline{R_k}) \tag{43}$$

for the pure imaginary values of α and β. The conditions (42) imply

$$\overline{\eta_k} = -\eta_k \qquad \overline{\xi_k} = \xi_k. \tag{44}$$

In this case matrices M, $\tilde{M}$ are real and as a consequence the solutions V, U, φ and q, p of the systems (3)–(5) and (8), (10) are real. Conditions (43) lead to the relation

$$\overline{\xi_k} = R_k \eta_{k\xi} \tag{45}$$

from which follows $\tilde{M} = \overline{M}$, $p = \epsilon\overline{q}$ and from formulae (38), (40), (41), (45) one can obtain for solution q of equation (9) the expression:

$$q = \det(1 + C) \tag{46}$$

where

$$C_{kn} = \sum_{l=1}^{N} \eta_k A_{nl}^{-1} R_l \overline{\eta}_{l\xi} \qquad A_{nl} = \delta_{nl} - \epsilon R_n \partial_\eta^{-1}(\overline{\eta}_n \eta_l). \tag{47}$$

From definitions (34) and (37) it is easy to see that the arbitrary functions ξ_k, η_k are the solutions of both equations of the type

$$X_{\xi\eta} = \epsilon X \qquad \alpha X_{\xi\xi} + \beta X_{\eta\eta} + X_t = 0. \tag{48}$$

4. Line solitons

Real-valued line solitons of systems (3)–(5) and (8), (10) in the case of real α and β correspond to the choice

$$f_k(\lambda, \overline{\lambda}) = \pi R_k \delta(\lambda - \mathrm{i}\beta_k) \qquad g_k(\lambda, \overline{\lambda}) = \pi \delta(\lambda - \mathrm{i}\alpha_k) \tag{49}$$

where R_k, α_k and β_k are arbitrary real constants. In this case due to (34), (37) and (49),

$$\xi_k(\xi, \eta, t) = -R_k \mathrm{e}^{F(\mathrm{i}\beta_k)} \qquad \eta_k(\xi, \eta, t) = -\frac{1}{\alpha_k} \mathrm{e}^{-F_k(\mathrm{i}\alpha_k)}, \tag{50}$$

and the solutions of systems (3)–(5) and (8), (10) are given by formulae (39)–(41) with matrix A of the form:

$$A_{nl} = \delta_{nl} - \partial_\eta^{-1}(\xi_{n\eta}\eta_l) = \delta_{nl} - \frac{R_n}{\alpha_l - \beta_n} exp[F(\mathrm{i}\beta_n) - F(\mathrm{i}\alpha_l)]. \tag{51}$$

The simplest solutions of such a type of equation corresponding to one term in sum (30) are

$$q = \frac{1-(\beta_1/\alpha_1)\phi}{1-\phi} \qquad p = \epsilon\frac{1-(\alpha_1/\beta_1)\phi}{1-\phi}$$
$$V = -\frac{q_\xi}{q} = -\frac{(\alpha_1-\beta_1)^2\phi}{\alpha_1(1-\phi)(1-(\beta_1/\alpha_1)\phi)}$$
$$U = -pq = -\epsilon\frac{(1-(\beta_1/\alpha_1)\phi)(1-(\alpha_1/\beta_1)\phi)}{(1-\phi)^2} \qquad \varphi = \ln 4U|_{\alpha=1,\beta=0} \tag{52}$$

where $\phi = R_1 e^{\Delta F}/(\alpha_1-\beta_1)$ and

$$\Delta F := F(i\beta_1) - F(i\alpha_1) = (\alpha_1-\beta_1)\left(\xi - \frac{\epsilon}{\alpha_1\beta_1}\eta\right) + (\alpha_1^2-\beta_1^2)\left(\alpha - \frac{\beta\epsilon^2}{\alpha_1^2\beta_1^2}\right)t.$$

Under the restriction $R_1/(\alpha_1-\beta_1) < 0$ the solutions $q,\ p$ of (8), (10) and under the restrictions $R_1/(\alpha_1-\beta_1) < 0$, $R_1\beta_1/(\alpha_1(\alpha_1-\beta_1)) < 0$ the solutions V, U and φ of (3)–(5) are non-singular and bounded line solitons. In terms of variable $\varphi = \ln 4U$ and for $\alpha = 1$, $\beta = 0$ one obtains from (39), (40) and (51), (52) corresponding solutions of 2D sinh–Gordon equation (5).

Complex-valued line solitons of systems (3), (4) and (8)–(10) in the case of pure imaginary α and β correspond to the choice

$$f_k(\lambda,\bar{\lambda}) = \pi R_k\delta(\lambda-\lambda_k) \qquad g_k(\lambda,\bar{\lambda}) = \pi\delta(\lambda-\overline{\lambda_k}) \tag{53}$$

where R_k are arbitrary real constants and $\lambda_k = \lambda_{kR} + i\lambda_{kI}$. In this case due to (34), (37) and (53)

$$\xi_k(\xi,\eta,t) = -R_k e^{F(\lambda_k)} \qquad \eta_k(\xi,\eta,t) = -\frac{i}{\overline{\lambda_k}}e^{-F(\overline{\lambda_k})} \tag{54}$$

and the solutions of (3), (4) and (8)–(10) corresponding to such choice of of kernel R are

$$V = -\frac{\partial}{\partial\xi}\ln\det(1+C) \qquad U = -\epsilon\det|1+C|^2$$
$$q = \det(1+C) \qquad p = \epsilon\bar{q} \tag{55}$$

where the matrix C is given by (47) and matrix A is

$$A_{kl} = \delta_{kl} - \epsilon R_k\partial_\eta^{-1}(\overline{\eta_k}\eta_l) = \delta_{kl} + \frac{iR_k e^{F(\lambda_k)-F(\overline{\lambda_l})}}{\lambda_k-\overline{\lambda_l}}. \tag{56}$$

The simplest solutions of (3), (4) and (8)–(10) of the type corresponding to one term $i\pi^2\delta(\mu-\lambda_1)\delta(\lambda-\overline{\lambda_1})$ in the sum (30) are given by the formulae:

$$V = \frac{4i\lambda_{1I}^2\phi}{(1+\phi)(1+(\lambda_1/\overline{\lambda_1})\phi)} \qquad U = -\epsilon\frac{|1+(\lambda_1/\overline{\lambda_1})\phi|^2}{(1+\phi)^2}$$
$$q = \frac{1+(\lambda_1/\overline{\lambda_1})\phi}{1+\phi} \qquad p = \epsilon\bar{q} \tag{57}$$

where $\lambda_1 = \lambda_{1R} + i\lambda_{1I}$, $\phi := (R_1/2\lambda_{1I})e^{\Delta F}$ and

$$\Delta F = F(\lambda_1) - F(\overline{\lambda_1}) = i(\lambda_1-\overline{\lambda_1})\left(\xi + \frac{\epsilon}{\lambda_1\overline{\lambda_1}}\eta\right) + (\lambda_1^2 - \overline{\lambda_1}^2)\left(\alpha - \beta\frac{\epsilon^2}{\lambda_1^2\overline{\lambda_1}^2}\right)t.$$

Under the restriction $R_1/2\lambda_{1I} > 0$ the complex-valued solutions (57) are non-singular bounded line solitons of corresponding equations. This type of line soliton with a constant asymptotic value at infinity for DS-I equation ($\sigma^2 = 1$) (9) was constructed by the Hirota method in [11], but the solutions in the present paper have a different parametrization.

5. Rational solutions

Rational solutions of integrable equations can also be easily constructed via $\bar\partial$-dressing method. Let us consider at first the case of real values of α and β in (1). For the reality condition (42) satisfies the following simple choice for the kernel R of $\bar\partial$-problem for example:

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \mathrm{i}\pi^2\sum_{k=1}^{N} S_k(\mu,\lambda)\delta(\mu-\mathrm{i}\alpha_k)\delta(\lambda-\mathrm{i}\alpha_k) \tag{58}$$

where $\delta(\mu-\alpha_k)$ is a complex Dirac function, $S_k(\mu,\lambda)$ some functions with the property $\overline{S_k(\mu,\lambda)} = S_k(-\overline{\mu},-\overline{\lambda})$ and $\alpha_1,\dots,\alpha_N$ is the set of real constants which are not equal to one another.

For this choice (58) of kernel R for the $\bar\partial$–problem (12) we give below detailed calculations for a prototype of such a calculation but for other choices of kernel R we shall only formulate the results in the following, omiting any details.

For the kernel R_0 of the form (58), one has from (12):

$$\frac{\partial\chi}{\partial\overline{\lambda}} = -\mathrm{i}\pi\sum_{k=1}^{N}\chi(\mathrm{i}\alpha_k \mathrm{e}^{F(\mathrm{i}\alpha_k)-F(\lambda)}S_k(\mathrm{i}\alpha_k,\lambda)\delta(\lambda-\mathrm{i}\alpha_k) \tag{59}$$

where $F(\lambda)$ is defined by (15). Then equation (26) gives at $\lambda\neq \mathrm{i}\alpha_k$, $(k=1,\dots,N)$:

$$\chi(\lambda,\overline{\lambda}) = 1-\mathrm{i}\sum_{k=1}^{N}\frac{\chi(\mathrm{i}\alpha_k)S_k(\mathrm{i}\alpha_k,\mathrm{i}\alpha_k)}{\lambda-\mathrm{i}\alpha_k} \tag{60}$$

while in the limits $\lambda\to \mathrm{i}\alpha_k$, using (26), (59), one gets for $\chi(\mathrm{i}\alpha_k)$, $(k=1,\dots,N)$:

$$\chi(i\alpha_k) = 1-\frac{1}{2}\iint_C\frac{\mathrm{d}\lambda\wedge\mathrm{d}\overline{\lambda}}{\lambda-\mathrm{i}\alpha_k}\sum_{l=1}^{N}\chi(\mathrm{i}\alpha_l)\mathrm{e}^{F(\mathrm{i}\alpha_l)-F(\lambda)}S_l(\mathrm{i}\alpha_l,\lambda)\delta(\lambda-\mathrm{i}\alpha_l). \tag{61}$$

The term in (61) with $l=k$ is equal to

$$\mathrm{i}\,\mathrm{Res}\,\frac{\chi(\mathrm{i}\alpha_k)\mathrm{e}^{F(\mathrm{i}\alpha_k)-F(\lambda)}S_k(\mathrm{i}\alpha_k,\lambda)}{(\lambda-\mathrm{i}\alpha_k)^2}\bigg|_{\lambda=\mathrm{i}\alpha_k} = \mathrm{i}\chi(\mathrm{i}\alpha_k)[S'(\mathrm{i}\alpha_k)-S_k(\mathrm{i}\alpha_k,\mathrm{i}\alpha_k)F'(\mathrm{i}\alpha_k)] \tag{62}$$

where

$$F(\mathrm{i}\alpha_k) := \frac{\partial F(\lambda)}{\partial\lambda}\bigg|_{\lambda=\mathrm{i}\alpha_k} = \mathrm{i}\left(\xi-\frac{\epsilon}{\alpha_k^2}\eta\right)+2\mathrm{i}\left(\alpha\alpha_k-\frac{\beta\epsilon^2}{\alpha_k^3}\right)t$$

and

$$S'(\mathrm{i}\alpha_k) := \frac{\partial S_k(\mathrm{i}\alpha_k,\lambda)}{\partial\lambda}\bigg|_{\lambda=\mathrm{i}\alpha_k} \qquad S_k := S(\mathrm{i}\alpha_k,\mathrm{i}\alpha_k).$$

As a result, equations (61) with different k give rise to the system:

$$\chi(\mathrm{i}\alpha_k)[1-\mathrm{i}S'(\mathrm{i}\alpha_k)+\mathrm{i}S_kF'(\mathrm{i}\alpha_k)]+\sum_{l\neq k}\frac{\chi(\mathrm{i}\alpha_l)S_l}{\alpha_k-\alpha_l} = 1. \tag{63}$$

For further calculations it is convenient to write the solution of system (63) in the form:

$$\chi(\mathrm{i}\alpha_k) = -\sum_{l=1}^{N}A_{kl}^{-1} \qquad (k=1,\dots,N) \tag{64}$$

where the $N \times N$ matrix A is

$$A_{kl} = d_k \delta_{kl} - \frac{1-\delta_{kl}}{\alpha_k - \alpha_l} \tag{65}$$

with

$$d_k := \xi - \frac{\epsilon}{\alpha_k^2}\eta + 2\left(\alpha\alpha_k - \frac{\beta\epsilon^2}{\alpha_k^3}\right) + \gamma_k \qquad \gamma_k := \frac{\mathrm{i}S'(\mathrm{i}\alpha_k) - 1}{S_k} = \overline{\gamma_k}.$$

From (60) one gets the coefficients $\tilde{\chi}_0$ and χ_{-1} of the series expansion of χ near the points $\lambda = 0$ and $\lambda = \infty$:

$$\tilde{\chi}_0 = 1 + \sum_{k=1}^{N} \frac{S_k}{\alpha_k}\chi(\mathrm{i}\alpha_k) \qquad \chi_{-1} = -\mathrm{i}\sum_{k=1}^{N} S_k \chi(\mathrm{i}\alpha_k). \tag{66}$$

Then using (25), (64)–(66) one obtains for q (the solution of (8), (10)) the following expression:

$$q = 1 - \sum_{k,l=1}^{N} \frac{1}{\alpha_k} A_{kl}^{-1} = 1 - \mathrm{tr}(BA^{-1}) = \det(1 - BA^{-1}) \tag{67}$$

where the $N \times N$ matrix B is

$$B_{pk} := \frac{1}{\alpha_k} \tag{68}$$

and the identity: $1 - \mathrm{tr}\, F = \det F$ was used for the matrix F of the first rank.

For the solution U of (3), (4) from (25), (64)–(66) one has:

$$U = -pq = -\epsilon + \sum_{k,l=1}^{N} (A_{kl}^{-1})_\eta = -\epsilon\left(1 - \sum_{k,p,l=1}^{N} A_{kp}^{-1}\frac{1}{\alpha_p^2}A_{pl}^{-1}\right). \tag{69}$$

With the use of identity

$$\sum_{k,p,l=1}^{N} A_{kp}^{-1}\frac{1}{\alpha_p^2}A_{pl}^{-1} = \sum_{p,q=1}^{N}\left(\frac{1}{\alpha_p}A_{pq}^{-1} - A_{pq}^{-1}\frac{1}{\alpha_q}\right) + \sum_{k,p,q,l=1}^{N} A_{kp}^{-1}\frac{1}{\alpha_p\alpha_q}A_{ql}^{-1} \tag{70}$$

which is valid for the matrix A defined by (65), the expression (69) for U can be transformed in the following way:

$$U = -\epsilon(1 - \mathrm{tr}\, BA^{-1})(1 + \mathrm{tr}\, A^{-1}B^T) = -\epsilon \det[(1 - BA^{-1})(1 + A^{-1}B^T)]. \tag{71}$$

Finally from (67) and (71) one finds for p

$$p = \epsilon \det(1 + A^{-1}B^T). \tag{72}$$

So, for the solutions p, q and V, U, φ of the systems (8), (10) and (3)–(5) we have the following formulae:

$$\begin{aligned} &q = \det(1 - BA^{-1}) \qquad p = \epsilon\det(1 + A^{-1}B^T) \\ &V = -\frac{\partial}{\partial\xi}\ln\det(1 - BA^{-1}) \qquad U = -\epsilon\det[(1 - BA^{-1})(1 + A^{-1}B^T)] \\ &\varphi = \ln 4U|_{\alpha=1,\beta=0}. \end{aligned} \tag{73}$$

The simplest solutions of the type which correspond to one term in the sum (58) have the form:

$$q = 1 - \frac{1/\alpha_1}{d_1} \qquad p = \epsilon\left(1 + \frac{1/\alpha_1}{d_1}\right)$$
$$V = -\frac{1}{\alpha_1 d_1(d_1 - 1/\alpha_1)} \qquad U = -\epsilon\left(1 - \frac{1/\alpha_1^2}{d_1^2}\right)$$
$$\varphi = \ln 4U|_{\alpha=1,\beta=0} \tag{74}$$

where

$$d_1 = \xi - \frac{\epsilon}{\alpha_1^2}\eta + 2\left(\alpha\alpha_1 - \frac{\beta\epsilon^2}{\alpha_1^3}\right)t + \gamma_1.$$

Analogous calculations can be made for the more complicated choice of the kernel R (α and β real constants) satisfying the reality condition (28):

$$R(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \mathrm{i}\pi^2\sum_{k=1}^{N}[S_k\delta(\mu-\lambda_k)\delta(\lambda-\lambda_k) + \overline{S_k}\delta(\mu+\overline{\lambda_k})\delta(\lambda+\overline{\lambda_k})]. \tag{75}$$

In this case it is convenient to introduce the sets Λ and Γ of complex constants Λ_k and Γ_k and the set X of quantities X_k, $(k = 1, \ldots, 2N)$:

$$\Lambda := (\Lambda_1 = \lambda_1, \ldots, \Lambda_N = \lambda_N; \Lambda_{N+1} = -\overline{\lambda_1}, \ldots, \Lambda_{2N} = -\overline{\lambda_N})$$
$$\Gamma := (\Gamma_1 = \gamma_1, \ldots, \Gamma_N = \gamma_N; \Gamma_{N+1} = \overline{\gamma_1}, \ldots, \Gamma_{2N} = \overline{\gamma_N}) \tag{76}$$
$$X := (X_1 = S_1\chi(\lambda_1), \ldots, X_N = S_N\chi(\lambda_N); X_{N+1} = \overline{S_1}\chi(-\overline{\lambda_1}), \ldots, X_{2N} = \overline{S_N}\chi(-\overline{\lambda_N})).$$

For the coefficients $\tilde{\chi}_0$ and χ_{-1} of the series expansions of χ near $\lambda = 0$ and $\lambda = \infty$ one obtains from (27), (75) the expressions:

$$\tilde{\chi}_0 = 1 + \mathrm{i}\sum_{k=1}^{2N}\frac{X_k}{\Lambda_k} \qquad \chi_{-1} = -\mathrm{i}\sum_{k=1}^{2N}X_k. \tag{77}$$

The system of equations for X_k has the form:

$$\sum_{l=1}^{2N}A_{kl}X_l = -1 \tag{78}$$

where the $2N \times 2N$ matrix A is

$$A_{kl} := d_k\delta_{kl} - \frac{\mathrm{i}(1-\delta_{kl})}{\Lambda_k - \Lambda_l} \tag{79}$$

with

$$d_k := \xi - \frac{\epsilon}{\Lambda_k^2}\eta - 2\mathrm{i}\left(\alpha\Lambda_k - \frac{\beta\epsilon^2}{\Lambda_k^3}\right)t + \Gamma_k.$$

The calculations in this case lead to the following expressions for the solutions p, q and V, U, φ of equations (8), (10) and (3)–(5) correspondingly:

$$q = \det(1 - \mathrm{i}BA^{-1}) = \overline{q} \qquad p = \epsilon\det(1 + \mathrm{i}A^{-1}B^T) = \overline{p}$$
$$V = -q_\xi/q = \overline{V} \qquad U = -pq = \overline{U} \qquad \varphi = \ln 4U|_{\alpha=1,\beta=0} \tag{80}$$

where the matrix A is given by (79) and $2N \times 2N$ matrix B is given by the formula: $B_{kl} := 1/\Lambda_l$. The simplest solutions which correspond to one term in the sum (75) have the form:

$$q = 1 - \mathrm{i}\left(\frac{d_1}{\lambda_1} - \frac{\overline{d_1}}{\overline{\lambda_1}}\right)\frac{1}{\Delta} - \frac{1}{|\lambda_1|^2\Delta} \qquad p = \epsilon\left(1 + \mathrm{i}\left(\frac{d_1}{\lambda_1} - \frac{\overline{d_1}}{\overline{\lambda_1}}\right)\frac{1}{\Delta} - \frac{1}{|\lambda_1|^2\Delta}\right)$$
$$V = -q_\xi/q = \overline{V} \qquad U = -pq = \overline{U} \qquad \varphi = \ln 4U|_{\alpha=1,\beta=0} \tag{81}$$

where

$$\Delta = |d_1|^2 - \frac{1}{4\lambda_{1R}^2} \qquad \text{and} \qquad d_1 = \xi - \frac{\epsilon}{\lambda_1^2}\eta - 2\mathrm{i}\left(\alpha\lambda_1 - \frac{\beta\epsilon^2}{\lambda_1^3}\right)t + \gamma_1.$$

After a change of variable $\varphi = \ln 4U$ one obtains from (73), (74) and (80), (81) (for $\alpha = 1$, $\beta = 0$) the corresponding solutions of 2D sinh–Gordon equation (5). It is easy to see that the rational solutions (73), (74), (80), (81) of systems (3)–(5) and (8), (10) obtained are singular.

One can make completely analogous calculations of rational solutions in the case of pure imaginary constants α and β in (1): $\alpha := \mathrm{i}\tilde{\alpha}$, $\beta := \mathrm{i}\tilde{\beta}$. In this case we only formulate final results. The reality condition (29) of U (or to the condition $p = \epsilon\overline{q}$) corresponds for example to the following simple choice of the kernel R of $\bar{\partial}$-problem:

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \mathrm{i}\pi^2 \sum_{k=1}^{N} S_k \delta(\mu - \alpha_k)\delta(\lambda - \alpha_k) \tag{82}$$

with $S_k = \overline{S_k}$ and $\alpha_k = \overline{\alpha_k}$.

For the coefficients $\tilde{\chi}_0$ and χ_{-1} of the series expansion of χ near $\lambda = 0$ and $\lambda = \infty$ one obtains from (27), (82) the expressions:

$$\tilde{\chi}_0 = 1 + \mathrm{i}\sum_{k=1}^{N} \frac{S_k \chi(\alpha_k)}{\alpha_k} \qquad \chi_{-1} = -\mathrm{i}\sum_{k=1}^{N} S_k \chi(\alpha_k). \tag{83}$$

The system of equations $\chi(\alpha_k)$ follows from (26), (82) and has the form:

$$\sum_{l=1}^{N} A_{kl}\chi(\alpha_l) = -1 \tag{84}$$

where

$$A_{kl} = d_k\delta_{kl} - \mathrm{i}\frac{1 - \delta_{kl}}{\alpha_k - \alpha_l} \tag{85}$$

with

$$d_k := \xi + \frac{\epsilon}{\alpha_k^2}\eta + 2\left(\tilde{\alpha}\alpha_k - \frac{\tilde{\beta}\epsilon^2}{\alpha_k^3}\right)t + \gamma_k$$

where γ_k are some real constants.

The use of (25), (83) and (84) leads to the following formulae for the exact solutions p, q and U, V of equations (8)–(10) and (3), (4) correspondingly:

$$q = \det(1 + \mathrm{i}BA^{-1}) \qquad p = \epsilon\,\det(1 - \mathrm{i}A^{-1}B^T) = \epsilon\overline{q}$$
$$V = -q_\xi/q \qquad U = -pq = \overline{U} \tag{86}$$

where matrix A is given by (85) and $B_{kl} := 1/\alpha_l$. The simplest solutions of this type which correspond to the one term in the sum (82) have the form:

$$q = 1 - \frac{\mathrm{i}/\alpha_1}{d_1} \qquad p = \epsilon\left(1 + \frac{\mathrm{i}/\alpha_1}{d_1}\right)$$
$$V = -\frac{\mathrm{i}}{\alpha_1 d_1(d_1 - \mathrm{i}/\alpha_1)} \qquad U = -pq = -\epsilon\left(1 + \frac{1/\alpha_1^2}{d_1^2}\right) \tag{87}$$

where

$$d_1 = \xi + \frac{\epsilon}{\alpha_1^2}\eta + 2\left(\tilde{\alpha}\alpha_1 - \frac{\tilde{\beta}\epsilon^2}{\alpha_1^3}\right)t + \gamma_1.$$

It is easy to see that solutions (86), (87) are singular.

One can satisfy the reality condition (29) by the more complicated choice of kernel R of $\bar{\partial}$-problem:

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \mathrm{i}\pi^2 \sum_{k=1}^{N} [S_k\delta(\mu - \lambda_k)\delta(\lambda - \lambda_k) + \overline{S_k}\delta(\mu - \overline{\lambda_k})\delta(\lambda - \overline{\lambda_k})]. \tag{88}$$

In this case it is convenient to introduce the sets Λ and Γ of complex constants Λ_k and Γ_k and the set of quantities X_k, $(k = 1, \ldots, 2N)$:

$$\Lambda := (\Lambda_1 = \lambda_1, \ldots, \Lambda_N = \lambda_N; \Lambda_{N+1} = \overline{\lambda_1}, \ldots, \Lambda_{2N} = \overline{\lambda_N})$$
$$\Gamma := (\Gamma_1 = \gamma_1, \ldots, \Gamma_N = \gamma_N; \Gamma_{N+1} = \overline{\gamma_1}, \ldots, \Gamma_{2N} = \overline{\gamma_N})$$
$$X := (X_1 = S_1\chi(\lambda_1), \ldots, X_N = S_N\chi(\lambda_N); X_{N+1} = \overline{S_1}\chi(\overline{\lambda_1}), \ldots, X_{2N} = \overline{S_N}\chi(\overline{\lambda_N})). \tag{89}$$

For the coefficients $\tilde{\chi}_0$ and χ_{-1} of the series expansion of χ near $\lambda = 0$ and $\lambda = \infty$ one obtains from (27) the expressions

$$\tilde{\chi}_0 = 1 + \mathrm{i}\sum_{k=1}^{2N} \frac{X_k}{\Lambda_k} \qquad \chi_{-1} = -\mathrm{i}\sum_{k=1}^{2N} X_k. \tag{90}$$

The system of equations for X_k has the form:

$$\sum_{l=1}^{2N} A_{kl}X_l = -1 \tag{91}$$

where

$$A_{kl} = d_k\delta_{kl} - \frac{\mathrm{i}(1 - \delta_{kl})}{\Lambda_k - \Lambda_l} \tag{92}$$

with

$$d_k := \xi - \frac{\epsilon}{\Lambda_k^2}\eta + 2\left(\tilde{\alpha}\Lambda_k - \frac{\tilde{\beta}\epsilon^2}{\Lambda_k^3}\right)t + \Gamma_k \qquad (k = 1, \ldots, 2N).$$

Then with the use of (25), (90)–(92) one easily calculates the solutions q, p and V, U of equations (8)–(10) and (3), (4) corresponding to the kernel (88):

$$q = \det(1 - \mathrm{i}BA^{-1}) \qquad p = \epsilon \det(1 + \mathrm{i}A^{-1}B^T) = \epsilon\overline{q}$$
$$V = -q_\xi/q \qquad U = -pq = -\epsilon|q|^2 \tag{93}$$

where $2N \times 2N$ matrix A is defined by (92) and $2N \times 2N$ matrix B is

$$B_{kl} := 1/\Lambda_l. \tag{94}$$

The simplest solutions of equations (3), (4) and (8)–(10) of this type which correspond to the one term in the sum (88) have the form:

$$q = 1 - \frac{\mathrm{i}}{\Delta}\left(\frac{d_1}{\overline{\lambda_1}} + \frac{\overline{d_1}}{\lambda_1} - \frac{\mathrm{i}}{|\lambda_1|^2}\right) \qquad p = \epsilon \overline{q}$$
$$V = -q_\xi / q \qquad U = -\epsilon |q|^2 \tag{95}$$

where

$$\Delta := |d_1|^2 + \frac{1}{4\lambda_{1I}^2} \qquad \text{and} \qquad d_1 = \xi - \frac{\epsilon}{\lambda_1^2}\eta + 2\left(\tilde{\alpha}\lambda_1 - \frac{\tilde{\beta}\epsilon^2}{\lambda_1^3}\right)t + \gamma_1$$

and λ_1 is the complex number $\lambda_1 = \lambda_{1R} + \mathrm{i}\lambda_{1I}$. As one can see from (91)–(95) the rational solutions q given by (93) and (95) are non-singular, bounded lump solutions of DS-I, DS-III equations with a constant asymptotic value at infinity. An example of this type of lump solutions for the DS-I equation was constructed by the Hirota method in [11], but the solutions in the present paper have different parametrization.

6. Conclusions

Let us make a few comments on the corresponding results of our and other papers with different approaches. Using the technique in the present paper, the $\bar{\partial}$-dressing method of Zakharov and Manakov [1–4], have constructed broad classes of exact solutions (non-singular and singular, real and complex) of equations (4), (5) and (8)–(10): solutions with functional parameters, multi-line solitons, rational solutions and, in particular, multi-lump solutions.

In the papers by Boiti *et al* [9] the simplest solutions of 2D sinh–Gordon equation (5) and a 2D dispersive long wave system of equations (4) with functional parameters and, in particular, one and two line soliton solutions via Backlund transformations were obtained. In the second paper [9] the IST scheme for the solution of the Cauchy problem for 2D dispersive long wave system of equations (4) was also developed. The solutions with functional parameters in the present paper are more general, and, in addition, we have obtained the rational and, in particular, the multi-lump solutions of the above mentioned equations.

All the constructions in the present paper are valid for $\sigma^2 = 1$ (see formulae (1)) and, in the case of DS system of equations (8), for DS-I ($\alpha = -\mathrm{i}$, $\beta = \mathrm{i}$) and DS-III ($\alpha = \mathrm{i}$, $\beta = \mathrm{i}$) equations and for the 2D system of nonlinear heat equations (α and β are real constants). The multi-line solitons, rational solutions and, in particular, multi-lump solutions of DS-I, DS-III equations constructed in this paper are very similar to those found in the paper by Satsuma and Ablowitz [11] by the Hirota method, but our solutions have a different parametrization.

In the recent papers by Guil and Manas [12] it has been shown that the DS system of equations (8) arises as the result of finite-rank constraints for the right-derivatives of certain automorphisms solving the heat equation. Using this fact the authors of papers [12] have constructed for DS-I, DS-II ($\sigma^2 = -1$) equations and 2D system of nonlinear heat equations the classes of exact solutions in the form of Wronskian and Grammian determinants, however the functional parameters of these solutions are different from those of the present paper. The relationship between the solutions with functional parameters in the present paper and those in [12] may be interesting and will be studied elsewhere.

Recently the DS-system of equations (8) has been considered in [13] where finite-gap solutions with several modifications of the DS-equations have been constructed via an algebraic geometric technique. It was shown in this paper that the finite-gap solutions include some classes of rational and soliton solutions which (such as those found in the present paper) have constant asymtotic values at infinity, these solutions also have a different parametrization from analogous solutions in the present paper; it may be interesting to study the relationship between the corresponding solutions.

In [14] the structure of explicit solutions of the DS-II equation ($\sigma^2 = -1$) has been studied with the use of the old method of Zakharov and Shabat [15]. It may be interesting to apply the more recent $\bar{\partial}$-dressing method of Zakharov and Manakov [1–4] to this type of equation—this will be done elsewhere.

Finally let us mention the paper [16] where the first linear spectral problem of the system (1) was considered and the IST scheme for this problem was developed via resolvent approach.

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